Limits of Recursive Sequences

44. Show that $a_1 = 2$

$$a_{n+1} = \frac{1}{3-a_n} \quad \text{satisfies } 0 < a_n \leq 2 \text{ and is decreasing.}$$

Find its limit.

- Using induction to show it's decreasing.

$$a_1 = 2 \rightsquigarrow a_2 = \frac{1}{3-2} = 1 \qquad a_2 < a_1 \checkmark$$

$$a_{k+2} = \frac{1}{3-a_{k+1}} < \underbrace{\frac{1}{3-a_k}}_{= a_{k+1}}$$

why? ↗

Since $a_{k+1} < a_k$,

$\Rightarrow \quad 3 - a_{k+1} > 3 - a_k$

and hence

$\Rightarrow \quad \dfrac{1}{3-a_{k+1}} < \dfrac{1}{3-a_k} \qquad a_{k+1} < a_k \checkmark$

Show it's bounded by $0 < a_n \leq 2$.

1st, its max is 2 since $\{a_n\}$ decreases from $a_1 = 2$.

2nd, suppose $a_k > 0$, then we need to show $a_{k+1} > 0$. $\Rightarrow$

$$a_{k+1} = \frac{1}{3-a_k} \overset{?}{>} \frac{1}{3-0}$$

If $a_k > 0$, then $\dfrac{1}{3-a_k} > \dfrac{1}{3}$. $\checkmark$

Limit

$$\lim_{n\to\infty} a_{n+1} = L = \frac{1}{3 - \lim_{n\to\infty} a_n}$$

$$\Rightarrow L = \frac{1}{3-L}$$

$$\Rightarrow -L^2 + 3L - 1 = 0$$

$$\Rightarrow L = \frac{-3 \pm \sqrt{9-4(-1)(-1)}}{2(-1)} = \frac{3 \pm \sqrt{5}}{2}$$

Choose the smaller one.

$$\boxed{L = \frac{3-\sqrt{5}}{2}}$$

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42. $\{a_n\}$ is given by $a_1 = \sqrt{2}$

$$a_{n+1} = \sqrt{2+a_n}$$

a) Show that $\{a_n\}$ is increasing and bounded above by 3. Deduce that $\lim_{n\to\infty} a_n$ exists.

- A monotonic and bounded sequence has a limit so we just need to show both is true. First, prove it's ~~bounded~~ increasing by induction.

$$a_1 = \sqrt{2} \Rightarrow a_2 = \sqrt{2+\sqrt{2}}$$

$$a_2 > a_1 \checkmark$$

$$a_{k+2} = \sqrt{2+a_{k+1}} \qquad a_{k+1} =$$

$$= \sqrt{2+\sqrt{2+a_k}} > \sqrt{2+a_k}$$

So, $a_{k+2} > a_{k+1}$, and so $\{a_n\}$ is increasing.

Now, let's show it's bounded. Suppose $a_k < 3$.

$$a_{k+1} = \sqrt{2+a_k} \overset{?}{<} \sqrt{2+3}$$

If $a_k < 3$, then so is a_{k+1}.

b) Find $\lim_{n\to\infty} a_n$.

$$L = \lim_{n\to\infty} a_{n+1} = \sqrt{2+\lim_{n\to\infty} a_n}$$

$$\Rightarrow L = \sqrt{2+L}$$

$$\Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow (L-2)(L+1) = 0$$

$$\Rightarrow \boxed{L=2} \text{ or } L=-1$$

- L needs to be positive

Limits of Recursive Sequences

34 a). If $\lim_{n\to\infty} a_n = L$, what is $\lim_{n\to\infty} a_{n+1}$?

Say $m = n+1$, then

$$\lim_{m-1\to\infty} a_m = \lim_{m\to\infty+1} a_m$$

$$= \lim_{m\to\infty} a_m$$ ← This is the same form as $\lim_{n\to\infty} a_n = L$

$$= L$$

c)

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{1}{1+a_n} \text{ for } n \geq 1 \end{cases}$$

What is the limit?

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{1+a_n}$$

$$\Rightarrow L = \frac{1}{1+\lim_{n\to\infty} a_n}$$

$$\Rightarrow L = \frac{1}{1+L}$$

$\Rightarrow L^2 + L - 1 = 0$ So $L = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{5}}{2} \Rightarrow \boxed{L = \dfrac{\sqrt{5}-1}{2}}$