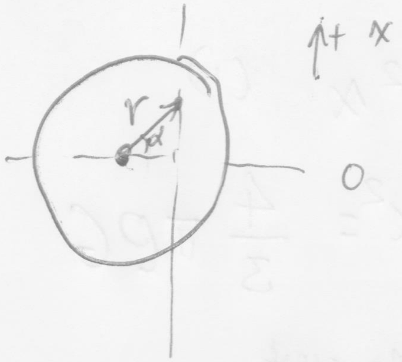
Gravity Train

Q) What is the time to reach other side? Train goes through a tunnel that goes through the Earth.

A)

$$x = \pm\, r\cos\alpha.$$



where r is distance from train to center of Earth

α is angle between middle of tunnel and vertical direction.

$$F = -\frac{Gm\,M(r)}{r^2}$$

Total mass depends on mass of Earth below the train. Note mass above train (outside r) all cancel each other out. No effect.

Let $M(r) = \left(\frac{4\pi r^3}{3}\right)\rho$

density

$$= -\frac{4}{3}\pi\rho G m r$$

Note this is net force but we only need F_x.

$$F_x = F\cos\alpha.$$

Let's do $\Sigma F = ma$

$$\Rightarrow m\ddot{x} = -\frac{4}{3}\pi\rho G m r\cos\alpha.$$

$$m\ddot{x} = -\frac{4}{3}\pi\rho G m x$$

2nd derivative of x.

This is a 2nd order differential eqn. It looks like a harmonic oscillator.

We ~~get~~ rewrite $\cancel{x(t)}$ = Divide by m:

$$\ddot{x} = -k^2 x \quad (1)$$

where $k^2 = \frac{4}{3}\pi\rho G$

Note, at surface,

$$F = -\frac{4}{3}\pi\rho G m R$$

radius of Earth

$$= -mg$$

$g \approx 9.8\,m/s^2$

So

$$\frac{g}{R} = k^2$$

$$g = +\frac{4}{3}\pi\rho G R$$

$\Rightarrow$ $x = A\cos\omega t$

$A = x_0$ is x at time t. It's the amplitude.

Note that we also know ω. Plug into original:

$$\dot{x} = -A\omega\sin\omega t$$

$$\ddot{x} = -A\omega^2\cos\omega t \rightarrow$$

Sub into (1):

$$-A\omega^2\cos\omega t = -k^2(A\cos\omega t)$$

$$\Rightarrow \omega^2 = k^2$$

$$\Rightarrow \omega = k.$$

Since ω is angular frequency, we can find the period:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{k}$$

However, we actually only need half of T for one way trip.

$$\frac{1}{2}T = \frac{\pi}{k} = \cancel{\pi\sqrt{\frac{R}{g}} = \pi\sqrt{9.8}}\ \pi\sqrt{\frac{R}{g}} = \pi\sqrt{\frac{6371\times10^3}{9.8}} = \boxed{2533 \text{ seconds}}$$